Natural Computation and Self-Organization:
The Physics of Information Processing in Complex Systems

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Homework 7

1. Calculating word probabilities from a model of a process: Consider the deterministic hidden Markov model of the Golden Mean process:

$$T^{(0)} = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}$$
$$T^{(1)} = \begin{pmatrix} 1/2 & 0 \\ 1 & 0 \end{pmatrix}$$

   (a) Find the equilibrium distribution $\langle e| = (p_A, p_B)$ for the internal states, A and B.

   (b) The probability of length $L = 1$ words is

$$p(0) = \langle e|T^{(0)}|n\rangle$$
$$p(1) = \langle e|T^{(1)}|n\rangle \ ,$$

   where $|n\rangle$ is the column vector containing all 1s. Calculate these.

   (c) Generally, the probability of word $w = s_0 s_1 s_2$ occuring is $p(w) = \langle e|T^{(w)}|n\rangle = \langle e|T^{(s_0)}T^{(s_1)}T^{(s_2)}|n\rangle$. Apply this technique to find the probability of length $L = 2$ words:

$$p(00) = \langle e|T^{(0)}T^{(0)}|n\rangle$$
$$p(01) = \langle e|T^{(0)}T^{(1)}|n\rangle$$
$$p(10) = \langle e|T^{(1)}T^{(0)}|n\rangle$$
$$p(11) = \langle e|T^{(1)}T^{(1)}|n\rangle \ .$$

2. For the Period-2, Golden Mean, and Even processes:

   (a) Calculate word probabilities for $L = 1, 2, 3, 4$.

   (b) Entropy growth: Find the block entropy $H(L)$ for each L above.

   (c) Entropy convergence: Find the length-L entropy rate $h_\mu(L)$ for each L above.

   (d) Find the asymptotic entropy rate h_μ using its closed-form expression:

$$h_\mu = -\sum_i p_i \sum_j \sum_s T^{(s)}_{i,j} \log_2 T^{(s)}_{i,j} \ .$$

   How does this compare with the $h_\mu(L)$ estimates?

   (e) Find the predictability gain $\Delta h_\mu(L)$ for each L above.

   (f) Approximate the total predictability **G** using the estimates from the previous step.

   (g) Calculate the redundancy $\mathcal{R}$. How does this compare with the previous estimate?

   (h) Use the estimates of h_μ and $h_\mu(L)$ to approximate the excess entropy **E** up to the given L.

   (i) Estimate the transient information **T** up to the given L.

(j) What do the values obtained above tell you about the processes under consideration?

3. The ϵ-machine for Golden Mean process:

   (a) Extend the topological reconstruction of the ϵ-machine for the Golden Mean process covered in the lecture to obtain the full probabilistic version. Show the parse tree and the morphs at a depth appropriate for correct reconstruction. Give the history sets for each causal state and the resulting ϵ-machine.

   (b) Calculate the entropy rate h_μ and the statistical complexity C_μ. Compare C_μ to the excess entropy $\mathbf{E}$. (You calculated the latter in the previous homework.)

4. The ϵ-machine for the Even process:

   (a) Reconstruct the ϵ-machine for the Even process. The correct answer is given in the Lecture Notes. Show the parse tree and the morphs at a depth appropriate for correct reconstruction.

   (b) Calculate the entropy rate h_μ and the statistical complexity C_μ. Compare the later to the excess entropy $\mathbf{E}$. (You calculated the latter in the previous homework, but this was only an approximation. You can look at the *RURO* paper on the Computational Mechanics Reader page at the course website for a more accurate value.)

Homework due one week after being assigned.